The 37th Science Olympiad for Young Mathematicians, Physicists and Chemists

January 2025
Rudbeckianska upper secondary school
Västerås, Sweden

Physics - Form 12

Question 1. Electron in cathode-ray tube (10 points)

The earliest known version of the cathode-ray tube is the Braun tube invented by Fredinand Braun as early as 1897. TV displays used the Braun tube to display images up until the early 2000s when it was replaced with LCD screens, see Figure 1. The cathode-ray tube consists of an electron gun and a deflector which guides the electron to the right position on the screen. In this problem we will consider a cathode-ray tube that uses two electrically charged plates to generate an electric field. The position where the electron hits the screen can be chosen by changing the electric field.

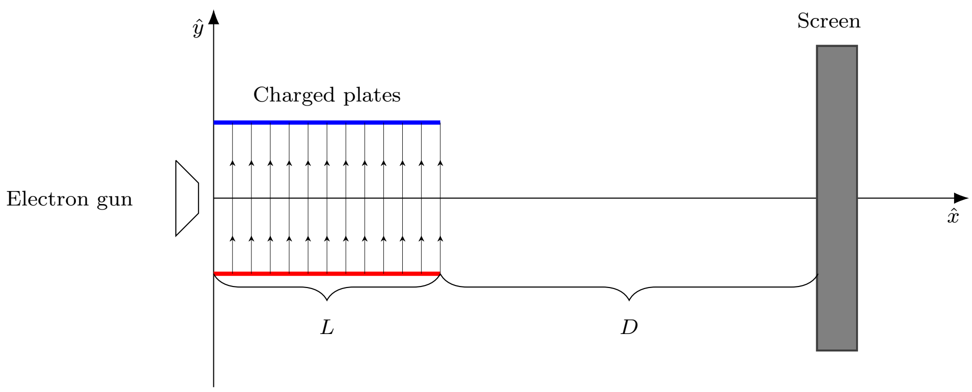

Figure 1: The cross section of the cathode-ray tube.

(a) In a cathode-ray tube, an electron of mass m_e is projected with velocity $\mathbf{v} = v_0\hat{x}$. The charged plates generate a constant electric field of $\mathbf{E} = E_0\hat{y}$. Where does the electron hit the screen? **Hint:** You should find the expression of the position in terms of the variables E_0, m_e, v_0, L and D. Neglect any other external forces and assume that the electron is at the origin at time t_0. (4)

(b) Determine where the electron will hit the screen if instead we have a changing electric field given by $\mathbf{E} = E_0 \sin(\omega t)\hat{y}$. Here ω is the angular frequency of the electric field and the electron enters the electric field at time $t_0 = 0$. **Hint:** You should find the expression of the position in terms of the variables E_0, m_e, v_0, L, D and ω. (6)

Solution:

(a) The force on the electron from the electric field is given by:

$$\mathbf{F}_E = -e\mathbf{E} = -eE_0\hat{y}$$

From Newton's second law we have $\mathbf{F_E} = m_e\mathbf{a}$. We can therefore find the acceleration of the electron inside the electric field.

$$\mathbf{a}_E = -\frac{eE_0}{m_e}\hat{y} \qquad \text{(1 point)}$$

We can integrate to obtain the velocity of the electron and use the initial condition of the electron, i.e. the velocity at $t = 0$.

$$\mathbf{v}_E = v_0\hat{x} - \frac{eE_0}{m_e}t\hat{y}$$

Integrating again we get the position of the electron.

$$\mathbf{x}_E = v_0t\hat{x} - \frac{eE_0}{m_e}t^2\hat{y}$$

The time spent inside the electric field is given by $t_1 = \frac{L}{v_0}$. The position and velocity of the electron when it leaves the electric field is thus (1 point for each):

$$\mathbf{v}_E = v_0\hat{x} - \frac{eE_0}{m_e}t_1\hat{y}$$
$$\mathbf{x}_E = L\hat{x} - \frac{eE_0}{m_e}\frac{t_1^2}{2}\hat{y}$$

After leaving the electric field no external forces acts on the electron $\mathbf{v_v} = v_0\hat{x} + v_y\hat{y}$ where $v_y = -\frac{eE_0}{m_e}t_1$.

$$\mathbf{x}_v = \int_{t_1}^{t_2} v_0\hat{x} + v_y\hat{y}\,dt \ = v_0(t_2 - t_1)\hat{x} - \frac{eE_0}{m_e}t_1^2(t_2 - t_1)\hat{y}$$

The time $t_2 - t_1$ is the time it takes the particle to travel from the end of the electric field to the screen, $t_2 - t_1 = \frac{D}{v_0}$.

Thus, we have $\mathbf{x}_v = D\hat{x} - \frac{eE_0}{m_e}t_1\frac{D}{v_0}\hat{y}$. This gives us the position on the screen where the electron hits to.

$$y_s = y_v + y_E = -\frac{eE_0}{m_e}t_1\frac{D}{v_0} - \frac{eE_0}{m_e}\frac{t_1^2}{2} = -\frac{eE_0}{m_e}t_1\left(\frac{D}{v_0} + \frac{t_1}{2}\right) = -\frac{eE_0L}{m_ev_0^2}\left(D + \frac{L}{2}\right)$$

(1 point for correct final expression).

(b)

$$\mathbf{F}_E = -e\mathbf{E} = -eE_0 \sin(\omega t)\hat{y}$$

$$\mathbf{a_E} = -\frac{eE_0}{m_e}\sin(\omega t)\hat{y}$$

Integrate to get $\mathbf{v_E}$ and $\mathbf{x_E}$ (1 point (without initial conditions)) with initial condition $\mathbf{v_E} = 0$ at $t_0 = 0$ in $\hat{y}$ direction (1 point for all initial conditions correct (v_E in each direction and x_E in each direction)).

$$\mathbf{v_E} = v_0\hat{x} + \frac{eE_0}{\omega m_e}(\cos(\omega t) - 1)\,\hat{y}$$

$$\mathbf{x_E} = v_0 t\hat{x} + \frac{eE_0}{\omega^2 m_e}(\sin(\omega t) - \omega t)\,\hat{y}$$

Time to reach the end of the electric field is

$$t_1 = \frac{L}{v_0}$$

The position in $\hat{y}$ direction at the end of the electric field is

$$y_E = \frac{eE_0}{\omega^2 m_e}\left(\sin\left(\frac{\omega L}{v_0}\right) - \frac{\omega L}{v_0}\right) \qquad \text{(1 point)}$$

and the velocity in $\hat{y}$ direction at the end of the electric field is

$$v_y = \frac{eE_0}{\omega m_e}\left(\cos\left(\frac{\omega L}{v_0}\right) - \frac{\omega L}{v_0}\right) \qquad \text{(1 point)}$$

Velocity and position after leaving the electric field is

$$\mathbf{v_v} = v_0\hat{x} + v_y\hat{y} \implies \mathbf{x_v} = v_0(t_2 - t_1)\hat{x} + v_y(t_2 - t_1)\hat{y}$$

where t_2 is the time to reach the screen

$$t_2 - t_1 = \frac{D}{v_0}$$

$$y_s = y_v + y_E = \frac{eE_0}{\omega m_e}\left(\frac{D\cos\left(\frac{\omega L}{v_0}\right) - L - D}{v_0} + \frac{\sin\left(\frac{\omega L}{v_0}\right)}{\omega}\right)$$

(2 points for correct final expression).

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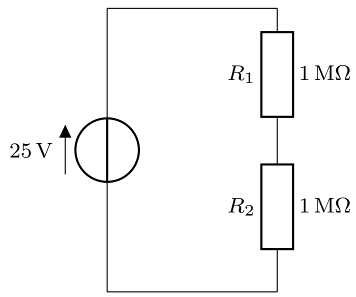
Question 2. Non-ideal electrical sources and meters **(10 points)**

(a) The voltage across the terminals of a black box is measured to $U_m = 25\,\mathrm{V}$ and the current to $I_m = 7.5\,\mathrm{mA}$. The measurements are made with an ideal voltmeter and an ideal ampere meter. The black box can be modeled, equivalently, as either a voltage source with voltage U_T and internal resistance R_T or as a current source with current I_N and internal resistance R_N. Calculate U_T, R_T, I_N, R_N and draw the equivalent circuits. (2)

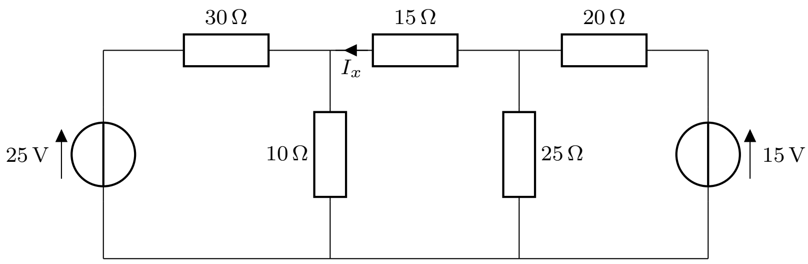
(b) The voltage over the resistor R_2 is measured with a non-ideal voltmeter with internal resistance $R_{\text{in}} = 10\,\mathrm{M\Omega}$. What voltage does the voltmeter display? (2)



(c) A voltmeter with internal resistance $R_{\text{in}} = 100\,\mathrm{k\Omega}$ can measure voltages up to $U_{\max} = 2\,\mathrm{V}$. Suggest a measuring technique that allows voltage measurements up to 230 V. (2)

(d) An ampere meter with internal resistance $R_{\text{in}} = 10\,\Omega$ can measure current up to $I_{\max} = 20\,\mathrm{mA}$. Suggest a measuring technique that allows current measurements up to 1 A. (2)

(e) Calculate the current I_x (2)



Solution:

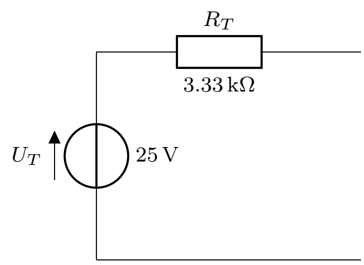
(a) There is does not run any current through an ideal voltmeter and for an ideal ampere meter there will be no voltage drop over it.

Voltage source

For a non-ideal voltage source there will be some internal resistance, i.e there will be some voltage drop inside the voltage source. This can be model by an resistor in series with an ideal voltage source, thus there will be voltage division between any load connected and the internal resistance of the voltage source.

When measuring the the voltage of the black box using an ideal voltmeter the current running through the voltmeter will be zero and therefore the current in the resistor R_T will be zero. Thus the voltage over R_T is zero and we have that $U_T = U_m = 25\,\mathrm{V}$.

Now we measure the current of the black box using an ideal ampere meter, the voltage drop over the ampere meter will be zero and thus the voltage over the resistor R_T will be $U_T = U_m$. To calculate the resistance R_T we can simply use ohms law to obtain $R_T = \frac{U_T}{I_m} = \frac{U_m}{I_m} \approx 3.33\,\mathrm{k\Omega}$.



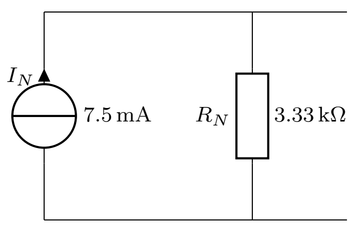
(1 point for correct circuit diagram and correct values R_T, U_T).

Current source

For a non-ideal current source there will similarly be internal resistance causing a drop in the current compared to an ideal current source, i.e we need a current divider to obtain this. A current divider is simply a resistor in parallel with the load.

Now to calculate I_N and R_N. We can begin by measuring the current over the black box using an ideal ampere meter. We now that there will be no voltage drop over the ideal ampere meter and therefore the voltage over the resistor R_N will be zero and the current through the resistor will thus also be zero. This means that $I_N = I_m = 7.5\,\mathrm{mA}$.

To calculate R_N we instead measure the voltage over the black box using an ideal voltmeter. An ideal voltmeter has no current running through it, thus the current over the resistor R_N will be $I_N = I_m$ and the resistance is thus calculated to $R_N = \frac{U_m}{I_m} = 3.33\,\mathrm{k\Omega}$.



(1 point for correct circuit diagram and correct values R_N, I_N)

(b) Non-ideal voltmeter can be viewed as a ideal voltmeter in parallel with a resistance R_{in} (Non-zero current through the voltmeter). Voltage divider

$$U_m = 25\,\mathrm{V} \cdot \frac{R_2 \parallel R_{\text{in}}}{R_1 + R_2 \parallel R_{\text{in}}} = 25\,\mathrm{V} \cdot \frac{\frac{10}{11}}{1 + \frac{10}{11}} = 25\,\mathrm{V} \cdot \frac{10}{21} \approx 11.9\,\mathrm{V}$$

(1 point for correct expression and 1 point for correct values).

(c) Known internal resistance $R_{\text{in}} = 100\,\mathrm{k\Omega}$ and max voltage $U_m = 2\,\mathrm{V}$. To allow measurements voltage up to $U'_m = 230\,\mathrm{V}$ place resistor in series with the voltmeter. The resistor R_x and the internal resistance creates a voltage divider

$$U_m = U_o \frac{R_{\text{in}}}{R_x + R_{\text{in}}} = U'_m \frac{R_{\text{in}}}{R_x + R_{\text{in}}} \qquad \text{(1 point)}$$

hence

$$R_x = R_{\text{in}} \left(\frac{U'_m}{U_m} - 1 \right) = 100\,\mathrm{k\Omega} \left(\frac{230\,\mathrm{V}}{2\,\mathrm{V}} - 1 \right) = 11.4\,\mathrm{M\Omega}$$

(1 point for correct value).

(d) Place a resistor R_y in parallel with the ampere meter, which creates a current divider

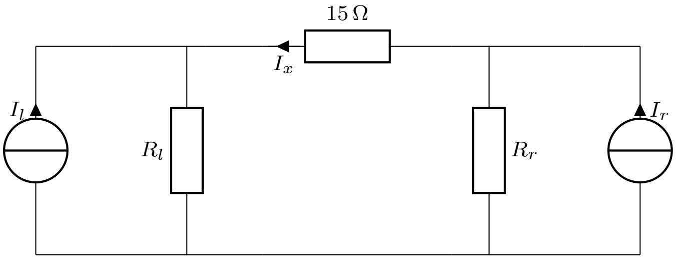
$$I_m = I_o \frac{R_y}{R_y + R_m} = I'_m \frac{R_y}{R_y + R_m} \qquad \text{(1 point)}$$

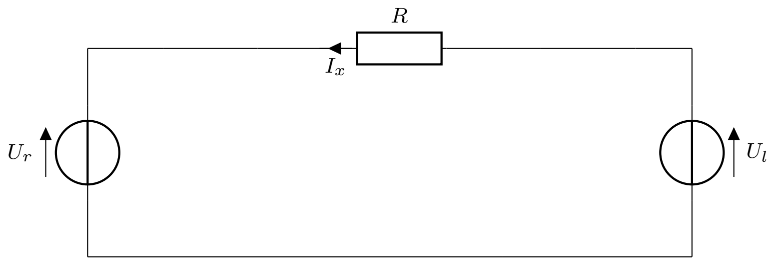
hence

$$R_y = \frac{I_m R_{\text{in}}}{I'_m - I_m} = \frac{20\,\mathrm{mA} \cdot 10\,\Omega}{1\,\mathrm{A} - 20\,\mathrm{mA}} = \frac{1}{49}\Omega \approx 200\,\mathrm{m\Omega}$$

(1 point for correct value).

(e) Switching back and forth between voltage and current sources (as seen in (a)) yields





with

$$\begin{aligned}
I_l &= \frac{25\,\mathrm{V}}{30\,\Omega} \approx 0.83\,\mathrm{A} \\
I_r &= \frac{15\,\mathrm{V}}{20\,\Omega} = 0.75\,\mathrm{A} \\
R_l &= 30\,\Omega \parallel 10\,\Omega = 7.5\,\Omega \\
R_r &= 20\,\Omega \parallel 25\,\Omega \approx 11\,\Omega \\
U_l &= R_l I_l = 6.25\,\mathrm{V} \\
U_r &= R_r I_r \approx 8.3\,\mathrm{V} \\
R_x &= R_l + 15\,\Omega + R_r \approx 33.6\,\Omega \\
I_x &= \frac{U_r - U_l}{R_x} \approx 62\,\mathrm{mA}
\end{aligned}$$

Using node analysis with node B to the left of $15\,\Omega$ resistor and node C to the right yields

$$0 = \frac{25 - U_B}{30} + \frac{U_C - U_B}{15} + \frac{0 - U_B}{10}$$
$$0 = \frac{U_B - U_C}{15} + \frac{15 - U_C}{20} + \frac{0 - U_C}{25}$$

hence

$$U_B\left(\frac{1}{30}+\frac{1}{15}+\frac{1}{10}\right)=\frac{25}{30}+\frac{U_C}{15}$$

and

$$U_C\left(\frac{1}{15}+\frac{1}{20}+\frac{1}{25}\right)=\frac{15}{20}+\frac{U_B}{15}.$$

Combining yields

$$U_B\left(\frac{1}{30}+\frac{1}{15}+\frac{1}{10}\right)=\frac{25}{30}+\frac{1}{15}\left(\frac{15}{20}+\frac{U_B}{15}\right)\left[\frac{1}{15}+\frac{1}{20}+\frac{1}{25}\right]^{-1}$$

$$U_B\left(\frac{1}{30}+\frac{1}{15}+\frac{1}{10}-\frac{1}{15^2}\left[\frac{1}{15}+\frac{1}{20}+\frac{1}{25}\right]^{-1}\right)=\frac{25}{30}+\frac{1}{15}\frac{15}{20}\left[\frac{1}{15}+\frac{1}{20}+\frac{1}{25}\right]^{-1}$$

$$U_B=\left(\frac{25}{30}+\frac{1}{15}\frac{15}{20}\left[\frac{1}{15}+\frac{1}{20}+\frac{1}{25}\right]^{-1}\right)\left(\frac{1}{30}+\frac{1}{15}+\frac{1}{10}-\frac{1}{15^2}\left[\frac{1}{15}+\frac{1}{20}+\frac{1}{25}\right]^{-1}\right)^{-1}$$
$$\approx 6.71\,\mathrm{V}$$

and

$$U_C=\left[\frac{15}{20}+\frac{U_B}{15}\right]\left(\frac{1}{15}+\frac{1}{20}+\frac{1}{25}\right)^{-1}\approx 7.64\,\mathrm{V}$$

hence

$$I_x=\frac{U_C-U_B}{15}\approx 62\,\mathrm{mA}$$

Can also be solved with mesh analysis with 3 meshes. (1 point for correct final expression (in terms of given quantities or quantities defined by the student) and 1 point for correct value).

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Question 3. Flood protection of Västerås **(10 points)**

During the summer of 2023 Svartån, the river that runs through Västerås, was flooded. During the autumn of the same year Västerås bought 200 meters of flood barriers. The city of Västerås wants to understand when and how these barriers might fail. Therefore, your task is to calculate the critical height of the water at which the barrier fails. The possible points of failure considered are that the barrier can either start to slide or tip.

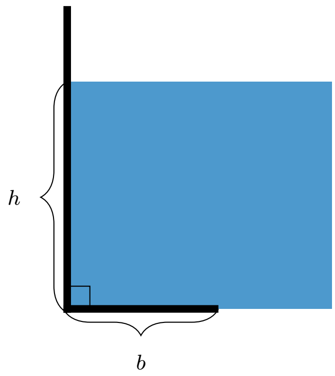


Figure 2: The cross section of the flood barrier.

A cross section of the flood barrier can be seen in Figure 2 and it has a total length of L. The coefficient of friction between the ground and the barrier can be assumed to be $\mu = 0.8$. Assume that the weight of the barrier is negligible in relation to the weight of the water and that there is a perfect seal between the ground and the barrier until it begins to tip. Also, assume that the barrier is a rigid body.

(a) Calculate the critical height of the water when the barrier starts to slide, that is, the relation between height h and length b? (4)

(b) Calculate the critical height of the water when the barrier starts to tip, that is, the relation between height h and length b? (4)

(c) How will the barrier fail and what is the critical height of the barrier? (2)

Solution:

(a)

$$p_b = \rho g h$$

Where p_b is the pressure on the bottom surface.

The pressure on the side can be written as:

$$p_s = \rho g(h - z)$$

Where $z = 0$ is at the bottom surface and positive z direction points upwards. The force on the bottom plate is given by.

$$F_b = p_b A_b = \rho g h b L$$

Thus the force due to friction is given by

$$F_f = F_b \mu = \mu \rho g h b L \qquad \text{(1 point)}$$

A small force element on the side is given by $p_s \cdot \Delta A = p_s \cdot L \cdot \Delta z = \rho g(h - z) \cdot L \Delta z$

Letting $\Delta z \to 0$ and summing over every force element acting on the side we get (1 point for correct expression F_s):

$$F_s = \int_0^h \rho g(h - z) L \, dz \; = \left[\rho g L(h \cdot z - \frac{z^2}{2})\right]_0^h = \rho g L h^2 - \frac{h^2}{2} = \frac{\rho g L h^2}{2}$$

Thus right before it starts sliding we have $F_f = F_s$ (1 point). Thus we get:

$$\frac{\rho g L h^2}{2} = \mu \rho g h b L \implies h = 2b\mu$$

Answer: At $h = 2b\mu = 1.6b$. (1 point)

This problem can also be solved by looking at the average force on the side of the barrier give instead of integrating.

(b) We calculate the moment of force around the corner of the barrier. For the bottom surface the force is constant and we can thus assume that the force acts on the midpoint, i.e. $M_b = F_b \frac{b}{2} = \frac{\rho g h L b}{2}$ (1 point).

For the moment of force on the side we are forced to integrate (1 point)

$$M_s = \int_0^h z p_s L \, dz \; = \int_0^h \rho g z(h - z) L \, dz \; = \rho g L \left[\frac{z^2 h}{2} - \frac{z^3}{3}\right]_0^h = \rho g L \frac{h^3}{6}$$

The critical height can be found at moment equilibrium (1 point):

$$M_s = M_b \implies \rho g L \frac{h^3}{6} = \rho g L \frac{hb^2}{2} \implies h = \sqrt{3}b$$

(1 point for correct final expression)

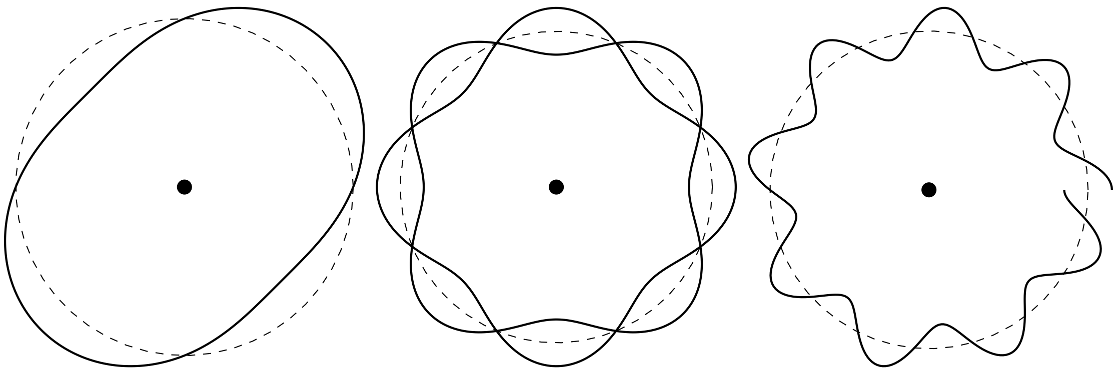
(c) The height it starts sliding is $h_s = 2b\mu = 1.6b$ and the height it starts tipping is at $h_t = \sqrt{3}b \approx 1.7b$. $h_s < h_t$. The failure of the barrier will be sliding (1 point with correct argument) and it will happen at the water height of $h = 1.6b$ (1 point).

Question 4. Wave-duality of particles and the quantization of hydrogen atoms (10 points)

De Broglie postulated in 1924 the wave-duality of matter and suggested that all particles could be considered waves with wavelength

$$\lambda = \frac{h}{p}$$

where p is the linear momentum of the particle. He suggested that the electron's orbit around the nucleus could be thought of as standing waves, see Figure 3.



(a) Standing wave

(b) Standing wave with lower wavelength than (a)

(c) Destructive interference

Figure 3: Standing stationary waves and destructive interference

In 1913 Bohr postulated that electrons orbit the nucleus in stable circular orbits and that the angular momentum of electrons are quantized (discrete/non-continuous values). De Broglie's condition of standing waves can be seen as an interpretation of Bohr's original postulate.

Neglect any gravitational effects and assume that the mass of the nucleus is much larger than that of the electron ($m_p \gg m_e$) such that the nucleus is stationary. The electron can be treated as a particle orbiting the nucleus; however, when viewed as a wave, the electron must still fulfill the condition of a standing wave. The charge of the nucleus is $q_p = e$ and the charge of the electron $q_e = -e$.

(a) Show that de Broglie's condition of standing waves is equivalent to Bohr's original postulate of quantized angular momentum. (2)

(b) Show that the condition of standing waves leads to quantized radii of the electron orbits and derive an expression for r_n. (4)

(c) Derive an expression for the energy levels E_n corresponding to the quantized radii. (4)

Hint: The reduced Planck's constant $\hbar = \frac{h}{2\pi}$ might be useful to simplify notation.

Solution:

(a) The condition of standing waves can be written as $2\pi r = n\lambda$ for $\lambda \in \mathbb{N}^+$ (1 point) (half wavelength does not work since we get *destructive interference* the second time around the nucleus, see (c) in Figure 3). The radius is thus quantized with $r = \frac{n\lambda}{2\pi}$. The linear momentum $p = \frac{h}{\lambda}$. The angular momentum is quantized with $L = mvr = pr = \frac{nh}{2\pi} = n\hbar$ (1 point).

(b) Circular orbit so electrostatic attraction equals the centripetal force

$$\frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r} \qquad \text{(1 point)}$$

and with $r = \frac{n\hbar}{mv}$

$$\frac{e^2}{(4\pi\varepsilon_0)n\hbar} = \frac{mv^2 r}{n\hbar} = v$$

hence

$$r_n = \frac{(4\pi\varepsilon_0)n^2\hbar^2}{e^2 m}$$

(3 points for correct final expression)

(c) Kinetic energy

$$T_n = \frac{1}{2}mv^2 = \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{n^2}$$

and potential energy

$$V_n = -\frac{e^2}{(4\pi\varepsilon_0)r} = -\frac{m}{\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{n^2} \qquad \text{(If at least one of } V_n \text{ or } T_n \text{ is correct give 1 point)}$$

hence the total energy

$$E_n = T_n + V_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{n^2}$$

(3 points for correct final expression).

(If the final expression in part (b) and (c) uses the *incorrect* half wavelength condition (which gives a factor 2 at each $\hbar$) but is otherwise correct give 1 point instead).