Mathematics – Form 11

Each full solution is worth 10 points. Just answers will not suffice to obtain full points, a thorough justification is always expected!

1

Find four prime numbers less than 100 which are factors of $3^{64} - 2^{64}$.

Solution. In this problem, the formula for the difference of squares comes handy:

$$\begin{aligned}
3^{64} - 2^{64} &= \left(3^{32} + 2^{32}\right)\left(3^{32} - 2^{32}\right) = \left(3^{32} + 2^{32}\right)\left(3^{16} + 2^{16}\right)\left(3^{16} - 2^{16}\right) = \\
&= \left(3^{32} + 2^{32}\right)\left(3^{16} + 2^{16}\right)\left(3^{8} + 2^{8}\right)\left(3^{8} - 2^{8}\right) = \\
&= \left(3^{32} + 2^{32}\right)\left(3^{16} + 2^{16}\right)\left(3^{8} + 2^{8}\right)\left(3^{4} + 2^{4}\right)\left(3^{4} - 2^{4}\right) = \\
&= \left(3^{32} + 2^{32}\right)\left(3^{16} + 2^{16}\right)\left(3^{8} + 2^{8}\right)\left(3^{4} + 2^{4}\right)\left(3^{2} + 2^{2}\right)\left(3^{2} - 2^{2}\right) = \\
&= \left(3^{32} + 2^{32}\right)\left(3^{16} + 2^{16}\right)\left(3^{8} + 2^{8}\right) \cdot 97 \cdot 13 \cdot 5
\end{aligned}$$

5, 13, and 97 are three prime factors of $3^{64}-2^{64}$. The fourth prime factor can be obtained from $3^8+2^8 = 6817$, as $6817 = 17 \cdot 401$ and 17 is a prime number.

Answer. Four prime factors of $3^{64} - 2^{64}$ that are less than 100 are: 5, 13, 17, and 97.

2

A game starts with four heaps of coins, containing 20, 22, 24 and 26 coins. Tom and Jerry move alternately. A move consists of taking either one coin from a heap, provided at least two coins are left behind in that heap, or a complete heap of two or three coins. The player who takes the last heap wins. Who wins if Tom makes the first move? Give a winning strategy.

Answer. Jerry wins.

Solution. There are $20 + 22 + 24 + 26 = 92$ coins in total (even number). Jerry wins if after his turn the total number of coins remains even. If Tom takes one coin then the total number of coins is odd. Jerry must then take either 1 or 3 coins from a heap, while avoiding the possibility of Tom taking 2 coins from a heap (if there are 3 coins in a heap then Jerry takes all of them, otherwise it's impossible for Tom to take 2 coins from a heap).

If Jerry applies this strategy then it takes 18 moves to remove all coins from the first heap. It takes $20, 22, 24$ moves to remove all coins from the second, third and fourth heap, respectively. Therefore the winner is chosen after $18 + 20 + 22 + 24 = 84$ moves – an even number of moves which suggests that Jerry wins.

3

Find all integers k such that the fraction $\dfrac{6k^2 + 5k + 4}{2k - 3}$ can be reduced to a positive integer.

Solution. After factoring the numerator we obtain

$$\frac{6k^2 + 5k + 4}{2k - 3} = \frac{3k\,(2k-3) + 14k + 4}{2k - 3} = \frac{3k\,(2k-3) + 7\,(2k-3) + 25}{2k - 3} = 3k + 7 + \frac{25}{2k-3}.$$

The expression $\dfrac{25}{2k-3}$ is an integer if $2k - 3$ is a factor of 25, i.e $\pm1, \pm5, \pm25$. This gives us that k can be $-11, -1, 1, 2, 4, 14$.

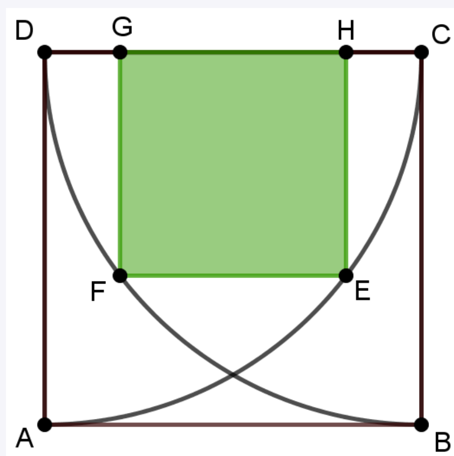
- If $k = -11$ then $3k + 7 + \dfrac{25}{2k-3} = -27$.
- If $k = -1$ then $3k + 7 + \dfrac{25}{2k-3} = -1$.
- If $k = 1$ then $3k + 7 + \dfrac{25}{2k-3} = -15$.
- If $k = 2$ then $3k + 7 + \dfrac{25}{2k-3} = 38$.
- If $k = 4$ then $3k + 7 + \dfrac{25}{2k-3} = 24$.
- If $k = 14$ then $3k + 7 + \dfrac{25}{2k-3} = 50$.

Therefore $\dfrac{6k^2+5k+4}{2k-3}$ can be reduced to a positive integer if $k = 2$, $k = 4$ or $k = 14$.

Answer. $k = 2$, $k = 4$ or $k = 14$.

4

Two identical quarter-circles are drawn inside the square $ABCD$. Another square, $EFGH$, is drawn inside that square so that E and F lie on quarter-circles and GH lies on CD. Find the area of $EFGH$ if $|AB| = 5$.



Solution. Let the side length of a square be x and due to symmetry $|DG| = |CH| = a$. Therefore $2a + x = 5 \implies a = \dfrac{5-x}{2}$ as it's the radius of the quarter-circle.

CFG is a right triangle with a hypotenuse of 5 and catheti of x and $a + x$. Therefore, after applying the Pythagorean theorem, we obtain

$$x^2 + (a+x)^2 = 5^2$$
$$x^2 + \left(\frac{5-x}{2} + x\right)^2 = 25$$

After simplifying this gives us

$$x^2 + 2x - 15 = 0$$

The roots of this quadratic equation are $x_1 = 3$ and $x_2 = -5$. As the side length of a square cant be negative then $x = 3$. Therefore the are of $EFGH$ is $3^2 = 9$ square units.

Answer. The area of $EFGH$ is 9 square units.

5

Let $f(x) = x^2 + px + q$ be a second degree polynomial with roots x_1 and x_2. Express $x_1^3 + x_2^3$ in terms of p and q.

Solution. From Vieta's formulas we know that $x_1 + x_2 = -p$ and $x_1 \cdot x_2 = q$.

$$x_1^3 + x_2^3 = (x_1 + x_2)\left(x_1^2 - x_1 x_2 + x_2^2\right) = (x_1 + x_2)\left(x_1^2 + 2x_1 x_2 + x_2^2 - 3x_1 x_2\right) = (x_1 + x_2)\left[(x_1 + x_2)^2 - 3x_1 x_2\right] =$$
$$= -p \cdot \left[(-p)^2 - q\right] = -p\left(p^2 - 3q\right) = -p^3 + 3pq$$

Answer. $x_1^3 + x_2^3 = -p^3 + 3pq$.