Mathematics Form 10

1. (The problem is believed to be supported by the teacher Tuomas Antero)
Let n be a two-digit number such that the square of the sum of digits of n is equal to the sum of digits of n^2.
(A) Give 3 examples of such two-digit numbers;
(B) Find the greatest such two-digit number;
(C) Find the sum all such two-digit numbers.

Solution

For any natural number N denote by $S(N)$ the sum of its digits
(e.g., $S(2014) = 2 + 0 + 1 + 4 = 7$) and note that for any 2-digit number N the number $S(N^2)$ is at most 35 (this is so because the only 4-digit number with the sum of digits 36, which is 9999, actually is not a square of an integer.
Hence is enough to check the 2-digit numbers with the digits a and b, or $10a + b$ with

$$a+b \leq 5.$$

Those numbers are

10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 30, 31, 32, 40. 41 and 50.

Straightforward computations indicates that only

10, 11, 12, 13, 20, 21, 22, 30 and 31

satisfy the wanted condition $S(N^2) = S^2(N)$.
This means there are exactly nine such 2-digits numbers, the biggest of them being 31 and with 170 = 10 + 11 + 12 + 13 + 20 + 21 + 22 + 30 + 31 as the sum of them all.

Answer

(A) , 11, 12, 13, 20, 21, 22, 30 and 31; (B) 31 (C) 170.

2. The teachers Dzintars and Liliia both regard the following problem as not especially difficult. They have in mind the following one-sentence problem:
Is the number $\sqrt{3+\sqrt{8}} - \sqrt{3-\sqrt{8}}$ rational or it is irrational?

Solution

$\sqrt{3+\sqrt{8}} - \sqrt{3-\sqrt{8}} = \sqrt{(\sqrt{2})^2 + 2\sqrt{2} + 1} - \sqrt{(\sqrt{2})^2 - 2\sqrt{2} + 1} = \sqrt{(\sqrt{2}+1)^2} - \sqrt{(\sqrt{2}-1)^2} =$
$= (\sqrt{2}+1) - (\sqrt{2}-1) = 2$, hence the number in question is a rational number.

Answer

The number is rational.

3. The teacher Tamm doesn‘t think that the solving of the following one-sentence problem can‘t take more than half an hour. The problem consists of very few words ands asks whether is it true or not that $6\sqrt{3} + 5\sqrt{5} + 4\sqrt{7} > 30$.

Solution

The solution contains three identical steps:
$108 > 100 \Leftrightarrow \sqrt{108} > \sqrt{100} \Leftrightarrow 6\sqrt{3} > 10$
$125 > 100 \Leftrightarrow \sqrt{125} > \sqrt{100} \Leftrightarrow 5\sqrt{5} > 10$
$112 > 100 \Leftrightarrow \sqrt{112} > \sqrt{100} \Leftrightarrow 4\sqrt{7} > 10.$
Adding them we get the required equality or

$6\sqrt{3}+5\sqrt{5}+4\sqrt{7}>30.$

Answer

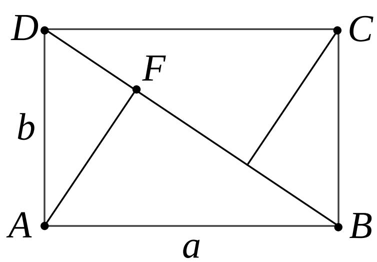
This is true.

4. (There exists an opinion that exactly this problem is recommended by Anna and Per Gunar).

The coefficients of the polynomial of the second degree $f(x)=ax^2+bx+c$ are known to be integers and, moreover, for any integer *n* the integer *f*(*n*) is divisible by 5. Prove that then all coefficients *a*, *b* and *c* of the polynomial *f*(*x*) are also divisible by 5.

5. Once, the teacher Anna and the principal Rima proposed to their students the following problem:

A rectangle *ABCD* has sides of length *a* and *b* where *b* < *a*. The lines through *A* and *C* perpendicular to the diagonal *BD* divide the diagonal into three segments of length 4, 5 and 4. Calculate $\frac{a}{b}$.



Solution

Denote by *F* the point, where the perpendicular through *A* to *DB* meets that diagonal *DB*. Then three right triangles *ADF*, *BDA* and *BAF* are all similar.

It is known that this similarity of right-angled triangles implies that $AF^2=DF\cdot FB$. In our case $AF^2=DF\cdot FB=4\cdot 9=36$, hence $AF=6.$ Applying the Theorem of Pythagoras to right-angled triangles *DFA* and *AFB* we get that the length *b* of the side *AD* is equal to $\sqrt{4^2+6^2}=\sqrt{52}=2\sqrt{13}$ and, similarly, the length *a* of the side *AB* is equal to $\sqrt{6^2+9^2}=\sqrt{117}=3\sqrt{13}$.

Finally, $\frac{a}{b}=\frac{3}{2}.$

Answer

$\frac{3}{2}.$